UK Junior Mathematical Olympiad 2005

Organised by The United Kingdom Mathematics Trust

Tuesday 14th June 2005

RULES AND GUIDELINES : READ THESE INSTRUCTIONS CAREFULLY BEFORE STARTING

1. Time allowed: 2 hours.
2. **The use of calculators** ***and measuring instruments*** **is forbidden.**
3. All candidates must be in *School Year 8 or below* (England and Wales), *S2 or below* (Scotland), *School Year 9 or below* (Northern Ireland).
4. For questions in Section A *only the answer is required*. Enter each answer neatly in the relevant box on the Front Sheet. Do not hand in rough work.

   For questions in Section B you must give *full written solutions*, including clear mathematical explanations as to why your method is correct.

   Solutions must be written neatly on A4 paper. Sheets must be STAPLED together in the top left corner with the Front Sheet on top.

   Do not hand in rough work.
5. Questions A1-A10 are relatively short questions. Try to complete Section A within the first 45 minutes so as to allow well over an hour for Section B.
6. Questions B1-B6 are longer questions requiring *full written solutions*.
   This means that each answer must be accompanied by clear explanations and proofs.
   Work in rough first, then set out your final solution with clear explanations of each step.
7. These problems are meant to be challenging! Do not hurry. Try the earlier questions in each section first (they tend to be easier). Try to finish whole questions even if you can't do many. A good candidate will have done most of Section A and given solutions to at least two questions in Section B.
8. Answers must be FULLY SIMPLIFIED, and EXACT using symbols like π, fractions, or square roots if appropriate, but NOT decimal approximations.

DO NOT OPEN THE PAPER UNTIL INSTRUCTED BY THE INVIGILATOR TO DO SO!

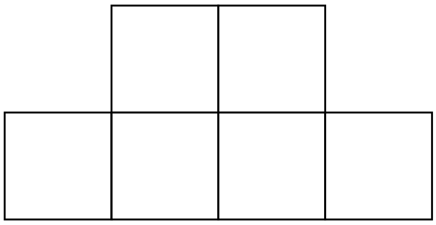
Section A

A1 How many seconds are there in one fortieth of an hour?

A2 The diagram shows a shape made from six squares, each of side 1cm.

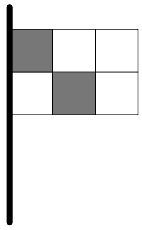
Four copies of the shape are placed together (without leaving any holes or having any overlaps) to form a rectangle.

What is the perimeter of the rectangle?



A3 Three different integers have a sum of 1 and a product of 36. What are they?

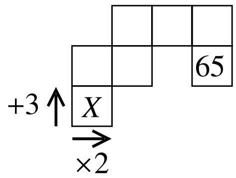
A4 A picture of a flag is to be completed by shading two squares which do not share an edge. The diagram shows one way in which this can be done.
How many different possible completed pictures are there (including the one shown)?



A5 In this puzzle, when you move up one square you **add 3**, when you move down one square you **subtract 3** and when you move to the right one square you **multiply by 2**.

The last square contains the number 65.

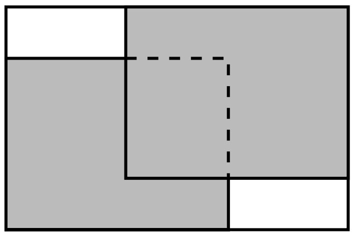
What number is in the square marked X ?



A6 Charlie's factory makes crème eggs and caramel eggs. The crème eggs are produced by a machine at the rate of 30 per minute, while the caramel eggs are produced by a different machine at the rate of 40 per minute. On a day when these two machines were in operation for a combined time of 18 hours, 36 000 eggs were produced in total. For how many hours was the crème egg machine in use?

A7 A sheet of paper is exactly the same size as a rectangular table top. The paper is cut in half and the two halves are placed on the table as shown.

What is the ratio of the area of table left uncovered (white) to the area which is covered twice?



A8 A large container holds 14 litres of a solution which is 25% antifreeze, the remainder being water. How many litres of antifreeze must be added to the container to make a solution which is 30% antifreeze?

A9 Colin has a collection of more than 24 coins. When he puts the coins in piles of 6, there are 3 coins remaining. When he puts the coins in piles of 8, there are 7 coins remaining. How many coins remain when he puts the coins in piles of 24?

A10 A closed rectangular box is a 'double cube', in which the top and bottom are squares, and the height is twice the width. The greatest distance between any two points of this box is 9 cm. What is the total surface area of the box?

Section B

Your solutions to Section B will have a major effect on the JMO results. Concentrate on one or two questions first and then **write out full solutions** (not just brief ‘answers’).

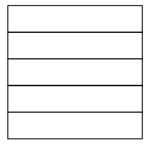
B1 The first three terms of a sequence are $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}$. The fourth term is $\frac{1}{2} - \frac{1}{3} + \frac{1}{4}$; henceforth, each new term is calculated by taking the previous term, subtracting the term before that, and then adding the term before that.

(i) Write down the first six terms of the sequence, giving your answers as simplified fractions.

(ii) Find the 10th term and the 100th term, and explain why they have to be what you claim.

B2 The diagram shows a square which has been divided into five congruent rectangles. The perimeter of each rectangle is 51 cm.

What is the perimeter of the square?



B3

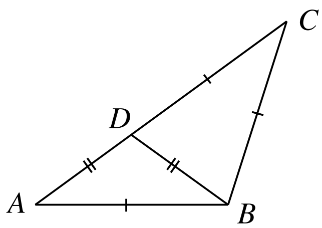
	175										70	

The diagram above is to be completed so that each box contains a whole number, the total of the numbers in the thirteen boxes is 2005 and the sum of the numbers in any three consecutive boxes is always the same.

In how many different ways is it possible to complete the diagram in this way?

B4 In this figure ADC is a straight line and $AB = BC = CD$. Also, $DA = DB$.

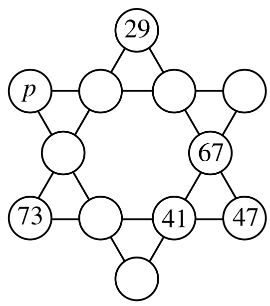
Find the size of $\angle BAC$.



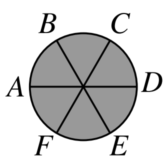
B5 In a magic hexagram, the numbers in every line of four circles have the same total. The diagram shows a magic hexagram which uses twelve different prime numbers.

Five numbers, including the smallest and the largest of the twelve primes, are shown.

Find the value of p, explaining the steps in your reasoning.



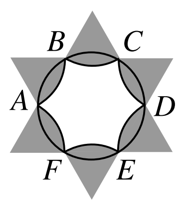
B6



Points A, B, C, D, E and F are equally spaced around a circle of radius 1. The circle is divided into six sectors as shown on the left.

The six sectors are then rearranged so that A, B, C, D, E and F lie on a new circle, also of radius 1, as shown on the right with the sectors pointing outwards.

Find the area of the curvy *unshaded* region.



UK Junior Mathematical Olympiad 2005 Solutions

A1 **90** One fortieth of an hour is one and a half minutes, that is 90 seconds.

A2 **20cm** When the four shapes are placed together, they will form a rectangle measuring 6cm by 4cm.

A3 **6, −3 , −2** As the sum of the integers is 1, it is clear that at least one of them is negative. Their product is positive so it may be deduced that exactly two of the integers are negative. Now we need to find three factors of 36 such that the largest is 1 greater than the sum of the other two. These are 6, 3 and 2.

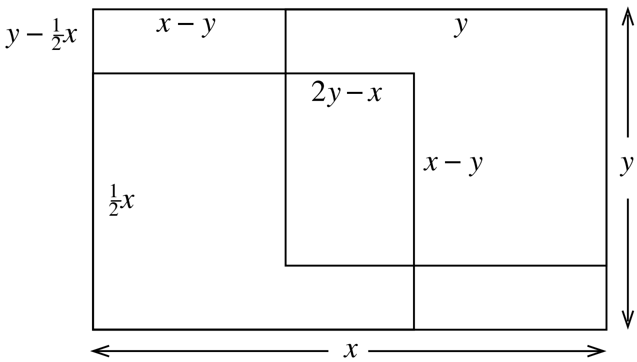
A4 **8** Number the squares 1 to 6 as shown. The possible pairings are 1 and 4, 1 and 5, 1 and 6, 2 and 3, 2 and 5, 2 and 6, 3 and 6, 4 and 5.

1	3	5
2	4	6

A5 **4** Working backwards from 65, we deduce that the numbers in the squares are 68, 34, 17, 14, 7, 4 respectively.

A6 **12** Let the crème egg machine be in use for x hours. Then $1800x + 2400(18 - x) = 36\,000$, that is $3x + 4(18 - x) = 60$, that is $72 - x = 60$.

A7 **1:1** Let the sheet of paper have length x and width y. Then the uncovered area consists of two congruent rectangles of length $x - y$ and width $y - \frac{1}{2}x$. So the uncovered area is $2(x - y)(y - \frac{1}{2}x)$, that is $(x - y)(2y - x)$.



The area covered twice is a rectangle of length $y - (x - y)$, that is $2y - x$, and width $\frac{1}{2}x - (y - \frac{1}{2}x)$, that is $(x - y)$. So the area covered twice is also $(x - y)(2y - x)$.

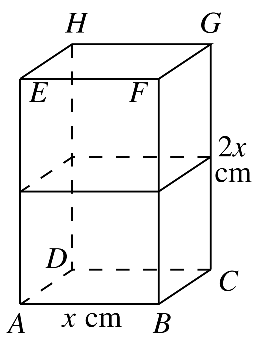
A8 **1** Let the extra volume of antifreeze required be x litres. Initially, there are 3.5 litres of antifreeze in the solution. So $3.5 + x = 0.3(14 + x)$, that is $35 + 10x = 42 + 3x$, so $x = 1$.

A9 **15** The number of coins in Colin's collection is 3 more than a multiple of 6 and also 7 more than a multiple of 8. The smallest number which satisfies both conditions is 15. The lowest common multiple of 6 and 8 is 24, so the conditions will also be met by numbers which exceed 15 by a multiple of 24, that is 39, 63, 87, etc. So when Colin puts his coins in piles of 24, 15 remain.

A10 **135cm²** Let the square base, $ABCD$, of the box be of side x cm. Then by Pythagoras' Theorem: $AC^2 = AB^2 + BC^2 = 2x^2$.

The greatest distance between any two points of the box is equal to the distance between opposite corners such as A and G. Applying Pythagoras' Theorem to triangle ACG: $AG^2 = AC^2 + CG^2 = 2x^2 + (2x)^2 = 6x^2$. So $6x^2 = 9^2 = 81$.

The total surface area, in cm², of the box $= 2 \times x^2 + 4 \times 2x^2 = 10x^2 = 10 \times \frac{81}{6} = 135$.



B1 (i) The fourth term = $\frac{1}{2} - \frac{1}{3} + \frac{1}{4} = \frac{5}{12}$. The fifth term = $\frac{5}{12} - \frac{1}{2} + \frac{1}{3} = \frac{1}{4}$. The sixth term = $\frac{1}{4} - \frac{5}{12} + \frac{1}{2} = \frac{1}{3}$. So the first six terms are $\frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{5}{12}, \frac{1}{4}, \frac{1}{3}$.

(ii) The seventh term = $\frac{1}{3} - \frac{1}{4} + \frac{5}{12} = \frac{1}{2}$. So the fifth, sixth and seventh terms equal the first three terms respectively and this means that the sequence will repeat itself every four terms since each term depends upon the previous three terms only.

So the tenth term equals the second term, $\frac{1}{3}$, and the hundredth term equals the fourth term, $\frac{5}{12}$.

B2 Let the length of the side of the square be $5x$ cm. Then each of the congruent rectangles has length $5x$ cm and width x cm, giving a perimeter of $12x$ cm. So $12x = 51$, that is $x = 4\frac{1}{4}$. The perimeter of the square = $20x$ cm, i.e. 85 cm.

B3

x	175	y									70	

Let the numbers in the first and third boxes be x and y respectively. Then the sum of the numbers in the first three boxes is $x + 175 + y$. However, this must equal the sum of the numbers in the second, third and fourth boxes and as the first two of these are 175 and y respectively, the number in the fourth box is x. Similarly, the sum of the numbers in the third, fourth and fifth boxes is $x + 175 + y$ and as the first two of these are y and x respectively, the number in the fifth box is 175. This argument may be continued to show that if the sum of the numbers in any three consecutive boxes is the same then the sequence of numbers must be x, 175, y, x, 175, y, x, 175, y, … and the required condition may be met in exactly one way or not at all.

As the number in the twelfth box is 70, we may deduce that y is 70. So the numbers in the thirteen boxes are x, 175, 70, x, 175, 70, x, 175, 70, x, 175, 70, x respectively and it remains to test if the value of x which makes the total of these numbers equal to 2005 is a whole number.

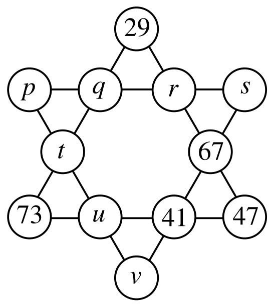
We require that $5x + 980 = 2005$, giving $x = 205$, so the diagram may be completed in exactly one way.

B4 Let $\angle BAC = x°$. Then $\angle BCA = x°$ since $AB = BC$. Also, $\angle ABD = x°$, since $DA = DB$. Furthermore, $\angle CDB = \angle DAB + \angle DBA = 2x°$ (exterior angle of a triangle).

As $BC = CD$, $\angle CBD = \angle CDB = 2x°$.

Triangle BCD has interior angles of $2x°$, $2x°$ and $x°$ so $5x = 180$ (angle sum of triangle), and so $\angle BAC = 36°$.

B5 The twelve prime numbers from 29 to 73 inclusive are 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73.



Let q, r, s, t, u, v represent the missing numbers in the circles shown.

Consider the two rows of four circles with one missing number each:

$73 + u + 41 + 47 = 47 + 67 + r + 29$ so $u = r - 18$. This means either that $r = 71$, $u = 53$ and each row totals 214 or $r = 61$, $u = 43$ and each row totals 204.

If the former is true, then the numbers still available are 31, 37, 43, 59, 61 and we require that $s + v = 214 - (41 + 67) = 106$. This is impossible.

If the latter is true, then the numbers still available are 31, 37, 53, 59, 71 and we require that $s + v = 204 - (41 + 67) = 96$. This is satisfied when s and v are 37 and 59 in some order.

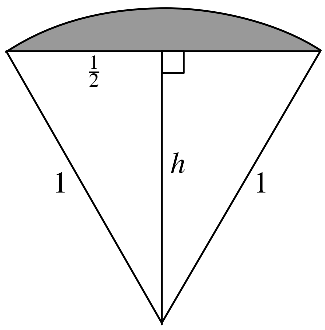
The numbers still available are 31, 53, 71.

If $s = 37$ and $v = 59$ then $p + q = 204 - (61 + 37) = 106$. This is impossible.

If $s = 59$ and $v = 37$ then $p + q = 204 - (61 + 59) = 84$. This is satisfied when p and q are 31 and 53 in some order. So t is 71 and we may deduce that $p = 204 - (71 + 43 + 37) = 53$.

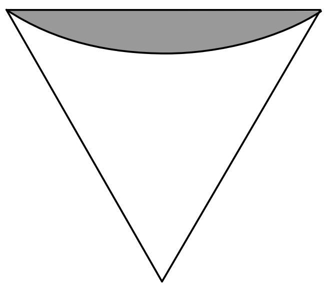
(We may check that $q = 31$ does give a total of 204 for the rows containing 53, 31, 61, 59 and 29, 31, 71, 73 respectively.)

B6 The symmetry of the figure means that the required area may be divided into six equal parts. The area of each of these parts is the area an equilateral triangle of side 1 minus the area of the shaded segment shown in the diagram.



Let the height of the equilateral triangle be h. Then, using Pythagoras' Theorem: $h^2 + \frac{1}{2}^2 = 1^2$ so $h = \sqrt{\frac{3}{4}} = \frac{1}{2}\sqrt{3}$ and the area of the triangle is $\frac{1}{2} \times \frac{1}{2}\sqrt{3} = \frac{1}{4}\sqrt{3}$.

Now the sector area is $\frac{1}{6} \times \pi \times 1^2$ so the area of the shaded segment is $\frac{1}{6}\pi - \frac{1}{4}\sqrt{3}$. Hence the area of the curvy unshaded region is .



$$6\left[\frac{\sqrt{3}}{4} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4}\right)\right] = 3\sqrt{3} - \pi.$$